Section 10.2

Sequences

Definition 10.2.1. Sequence
A *sequence* $\{a_n\}_{n=1}^{\infty}$ is an ordered list of numbers

$$\{a_1, a_2, a_3, \dots\}$$

with index n. A sequence may be generated by a recurrence relation of the form $a_{n+1} = f(a_n)$ for $n = 1, 2, 3, \dots$ where a_1 is some given number. A sequence may also be defined with an explicit formula $a_n = f(n)$, for $n = 1, 2, 3, \dots$

Example 10.2.2.

1. $\left\{\dfrac{1}{2^n}\right\}_{n=0}^{\infty}$

2. $\left\{\dfrac{(-1)^n n}{n^2+1}\right\}_{n=1}^{\infty}$

3. $\{a_n\}_{n=1}^{\infty}$ where $a_{n+1} = a_n + a_{n-1}$ and $a_0 = 0, a_1 = 1$.

Definition 10.2.3. Limit of a Sequence
If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases, that is for any error ε there is an integer N such that $|a_n - L| < \varepsilon$ for all $n > N$, then

$$\lim_{n\to\infty} a_n = L$$

and we say the sequence *converges* to L. If the terms do not approach a single number as n increase, then the sequence *diverges*.

You can think of a sequence converging to L as giving an approximation of L for large n. Limits of sequences are not that different than limits found in calculus 1.

Theorem 10.2.4. Limits of Sequences from Functions
Suppose f is a function such that $f(n) = a_n$, for positive integers n. If

$$\lim_{x\to\infty} f(x) = L$$

then

$$\lim_{n\to\infty} a_n = L.$$

We allow L to be $\pm\infty$, though in both cases the limits do not exist.

Example 10.2.5. Compute the limits of the following sequences.

1. $\left\{\dfrac{3n^3}{n^3+1}\right\}_{n=1}^{\infty}$

2. $\lim\limits_{n\to\infty}\left(\dfrac{n+5}{n}\right)^n$

3. $\lim_{n\to\infty} n^{1/n}$

Definition 10.2.6. Terminology for Sequences

- $\{a_n\}$ is *increasing* if $a_{n+1} > a_n$.
- $\{a_n\}$ is *nondecreasing* if $a_{n+1} \geq a_n$.
- $\{a_n\}$ is *decreasing* if $a_{n+1} < a_n$.
- $\{a_n\}$ is *nonincreasing* if $a_{n+1} \leq a_n$.
- $\{a_n\}$ is *monotonic* if it is either nonincreasing or nondecreasing.
- $\{a_n\}$ is *bounded above* if there is a number M such that $a_n \leq M$. The sequence is *bounded below* if there is a number N such that $a_n \geq N$.
- If $\{a_n\}$ is bounded above and bounded below, then we say $\{a_n\}$ is *bounded*.

Boundedness is useful for determining convergence.

Theorem 10.2.7. Bounded Monotonic Sequences
A bounded monotonic sequence converges.

Example 10.2.8. Observe the behavior of the following sequences

1. $\dfrac{n^{3/2}}{n^{3/2}+1}$

2. $\dfrac{(-1)^n n^{3/2}}{n^{3/2}+1}$

3. $d_{n+1} = 0.5d_n + 100$ with $d_1 = 200$

Theorem 10.2.9. Geometric Sequences
Geometric sequences have the property that each term is obtained by multiplying the previous term by a fixed ratio, i.e. $\{ar^n\}$.

$$\lim_{n\to\infty} ar^n = \begin{cases} 0 & \text{if } |r| < 1 \\ a & \text{if } |r| = 1 \\ \text{does not exist} & \text{if } r \leq 1 \text{ or } r > 1 \end{cases}$$

Example 10.2.10. Compute $\lim_{n\to\infty} \dfrac{2n^2+n}{2^n(3n^2-4)}$

Theorem 10.2.11. Squeeze Theorem for Sequences
Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$, for all integers n greater than some index N. If

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then

$$\lim_{n\to\infty} b_n = L.$$

Example 10.2.12. Compute $\lim_{n\to\infty} \dfrac{\cos n}{n^2+1}$

Growth Rates

Definition 10.2.13. Let $\{a_n\}, \{b_n\}$ be nondecreasing sequences. To compare the growth of $\{a_n\}$ and $\{b_n\}$ we evaluate $\lim_{n\to\infty} \dfrac{a_n}{b_n}$.
If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

then $\{b_n\}$ grows faster than $\{a_n\}$ and we write $\{a_n\} \ll \{b_n\}$.
If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$

then $\{a_n\}$ grows faster than $\{b_n\}$, $\{b_n\} \ll \{a_n\}$.

Example 10.2.14. Compare the growth of $\{b^n\}, \{n!\}$, and $\{n^n\}$.

Theorem 10.2.15. Growth Rates
The following sequences are ordered by increasing growth rates

$$\{\ln^q n\} \ll \{n^p\} \ll \{n^p \ln^r n\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}.$$

The ordering applies for all $p, q, r, s, b > 1$.

Example 10.2.16. Find the limit of the following sequences or determine that the sequence diverges.

1. $\left\{ \dfrac{n^3}{n^4+1} \right\}$

2. $\left\{ \arctan\left(\dfrac{10n}{10n+4} \right) \right\}$

3. $\left\{ \ln \sin \dfrac{1}{n} + \ln n \right\}$

4. $\left\{\dfrac{e^n + \pi^n}{e^n}\right\}$

5. $\{2^{n+1}3^{-n}\}$